



II Semester M.Sc. Degree Examination, July 2017
(CBCS)

MATHEMATICS
M205T : Functional Analysis

Time : 3 Hours

Max. Marks : 70

- Instructions** : 1) Answer any five full questions.
2) All questions carry equal marks.

1. a) State and prove
i) Cauchy-Schwartz inequality.
ii) Minkowski inequality.
- b) Show that the collection $B(X)$ of all bounded functions on a set X is a complete normed linear space.
- c) Show that if $T : N \rightarrow N'$ is a linear transformation between normed linear spaces N and N' then prove that the following are equivalent.
i) T is continuous at the origin.
ii) T is bounded. (5+4+5)
2. a) If M is a closed subspace of a normed linear space N then show that there is a natural map $T : N \rightarrow \frac{N}{M}$ which is linear and continuous with $\|T\| \leq 1$.
- b) State and prove Hahn Banach theorem for both real and complex cases. (4+10)
3. a) If N is a normed linear space and $x_0 \in N$ with $x_0 \neq 0$ then show that there exists a functional $f_0 \in N^*$ such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.
- b) If M is a closed linear subspace of a normed linear space N and $x_0 \notin M$ then show that there is a functional $h \in N'$ such that $h(x_0) = 1$ and $h(M) = 0$ with $\|h\| = \frac{1}{d}$ where $d = d(x_0, M)$. (7+7)



4. a) State the open mapping theorem. Use it to prove that a one-one linear continuous map T from a Banach space B onto a Banach space B' is a homeomorphism.
- b) State and prove closed graph theorem.
- c) Let N be a normed linear space and X be a non-empty subset of N . If $\|f(x)\|$ is bounded for each $f \in N'$ then show that X is bounded. (3+8+3)
5. a) Prove the Parallelogram law and polarisation identity in a Hilbert space.
- b) Show that orthogonal complement of a subset of a Hilbert space H is a closed linear subspace of H .
- c) Prove that translation on a Hilbert space is a homeomorphism. (5+4+5)
6. a) Let M be a closed linear subspace of a Hilbert space H and let $x \notin M$ and $d = d(x, M)$. Then prove that there is a unique vector $y_0 \in M$ such that $\|x - y_0\| = d$.
- b) If M is a closed linear subspace of a Hilbert space H then prove that $H = M \oplus M^\perp$.
- c) If $\{e_1, e_2, \dots, e_n\}$ is a finite orthonormal set in a Hilbert space H and $x \in H$ then prove that
- $$i) \sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2 \quad ii) x = \sum_{i=1}^n (x, e_i) e_i + (e_i, x)^\perp \quad (5+4+5)$$
7. a) If T is an operator on a Hilbert space H then prove that $T = 0$ if and only if $(Tx, y) = 0 \forall x, y \in H$.
- b) Show that the collection of all self adjoint operators on a Hilbert space H form a closed real linear subspace of H .
- c) Show that eigen values of a self adjoint operator are real and the eigen vectors associated with distinct eigen values are orthogonal. (3+4+7)
8. a) If N_1 and N_2 are two normal operators on a Hilbert space H such that either commutes with the adjoint of the other then prove that the product and the sum of N_1 and N_2 are also normal operators.
- b) Define an Unitary operator. Show that an Unitary operator preserves the inner product and the norm.
- c) Show that if a normal operator T has eigen value λ then its adjoint T^* has eigen value $\bar{\lambda}$. (5+6+3)